Constraints on Hadronic Spectral Functions From Continuous Families of Finite Energy Sum Rules

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Hadronic τ decay data is used to study the reliability of various finite energy sum rules (FESR's). For intermediate scales ($s_0 \sim 2-3 \, \text{GeV}^2$), those FESR's with weights s^k are found to have significant errors, whereas those with weights having a zero at the juncture ($s = s_0$) of the cut and circular part of the contour work very well. It is also shown that a combination of two such sum rules allows rather strong constraints to be placed on the hadronic spectral function without sacrificing the excellent agreement between OPE and hadronic representations. The method then is applied to the strangeness $S = -1$ and pseudoscalar isovector channels, where it is shown to provide novel constraints on those ansatze for the unmeasured continuum parts of the spectral functions employed in existing extractions of $m_u + m_d$, m_s. The continuum ansätz in the latter channel, in particular, is shown to produce a rather poor match to the corresponding OPE representation, hence re-opening the question of the value of the light quark mass combination $m_u + m_d$.

11.55.Hx,12.15.Ff,13.35.Dx

I. INTRODUCTION

As is well-known, for typical hadronic correlators $\Pi(s)$, analyticity, unitarity, and the Cauchy theorem imply the existence of dispersion relations which, due to asymptotic freedom, allow non-trivial input in the form of the operator product expansion (OPE) at large spacelike $s = q^2 = -Q^2$. The utility of these relations can be improved by Borel transformation [1–3], which introduces both an exponential weight, $exp(-s/M^2)$, (where M, the Borel mass, is a parameter of the transformation), on the hadronic (spectral integral) side and a factorial suppression of contributions from higher dimension operators on the OPE side. The exponentially decreasing weight allows rather crude approximations for the large s part of the spectral function to be tolerated. Typically, one employs essentially a "local duality" approximation, i.e., uses the OPE version of the spectral function for all s greater than some "continuum threshold". Competition between optimizing suppression of contributions from the crudely modelled "continuum" and convergence of the OPE usually results in a "stability window" in M for which neither contributions from the continuum, nor those from the highest dimension operators retained on the OPE side, are completely negligible. The resulting uncertainties have to be carefully monitored to determine the reliability of a given analysis [4]. The presence of the decreasing exponential weight also means that the method is less sensitive to the parameters of the higher resonances in the channel in question.

In this paper we investigate the alternative to Borel-transformed (SVZ) sum rules provided by those finite energy sum rules (FESR's) [5–9] generated by integration over the "Pac-man" contour (running from s_0 to threshold below the cut on the real timelike axis, back from threshold to s_0 above the cut, and closed by a circle of radius s_0 in the complex s plane). One advantage of such FESR's is the absence of an exponentially decreasing weight. Indeed, if $\Pi(s)$ is a hadronic correlator without kinematic singularities, and $w(s)$ any analytic weight function, then, with the spectral function $\rho(s)$ defined as usual,

$$\int_{s_{th}}^{s_0} w(s)\rho(s) \, ds = -\frac{1}{2\pi i} \int_{|s|=s_0} w(s)\Pi(s) \, ds \ . \tag{1}$$

Such FESR's have to date usually employed integer power weights, $w(s) = s^k$, $k = 0, 1, 2$ (see, *e.g.*, the recent extraction of $m_u + m_d$ in Ref. [10] using the isovector pseudoscalar correlator), but are, of course, valid for any $w(s)$ analytic in the region of the contour.

One interesting FESR involving a non-integer-power weight is that relevant to hadronic τ decay. Neglecting the tiny contributions proportional to $(m_d - m_u)^2$ in the isovector vector (IV) current correlator (which are, in any case, hard

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to handle reliably on the OPE side [11,12]), the ratio of the non-strange hadronic to electronic widths is proportional to

$$\int_{4m_\pi^2}^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2\frac{s}{m_\tau^2}\right) \rho_\tau^{(0+1)}(s) \tag{2}$$

with $\rho_\tau^{(0+1)}(s)$ the sum of longitudinal and transverse contributions to the corresponding spectral function [13]. This is the hadronic side of a FESR with weight

$$w_\tau(s) = \frac{1}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2\frac{s}{m_\tau^2}\right) , \tag{3}$$

the OPE side of which is

$$\frac{i}{2\pi} \oint_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2\frac{s}{m_\tau^2}\right) \Pi_{V,ud}^{(0+1)}(s) . \tag{4}$$

Eq. (4) thus provides an expression for the non-strange hadronic τ decay width [14–16] which requires as input only $a(m_\tau^2) = \alpha_s(m_\tau^2)/\pi$ and the relevant $D = 4$ and $D = 6$ condensates (these latter, in fact, give rather small contributions) [14–16]. This representation works extremely well, in the sense that the $a(m_\tau^2)$ value required by τ decay data (see Refs. [17,18] and earlier references cited therein) is nicely consistent, after running, with that measured experimentally at the Z mass scale [16]. One can also verify that the success of the underlying FESR is not a numerical accident by comparing the hadronic and OPE sides for a range of s_0 values $< m_\tau^2$ [19]. As shown by ALEPH, and reiterated below, the agreement between the two representations is excellent for all s_0 between 2 GeV^2 and m_τ^2.

The success of the τ decay FESR has a simple physical explanation. As argued in Ref. [20], for large enough s_0, the OPE should provide a good representation of $\Pi(s)$ over most of the circle $|s| = s_0$. When local duality is not yet valid, however, this representation will necessarily break down over some region near the timelike real axis. Since, with Δ some typical hadronic scale, the problematic region represents a fraction $\Delta/\pi s_0$ of the full circle, one might expect the error on the OPE side of a given FESR to be $\sim \Delta/\pi s_0$. Consider, however, a correlator having perturbative contribution of the form $Q^{2n} \left(1 + c_1 a(Q^2) + c_2 a(Q^2)^2 + \cdots\right)$, with n positive. Expanding this expression in terms of $a(s_0)$, one obtains $Q^{2n} \left(1 + c_1 a(s_0) + \mathcal{O}\left(a(s_0)\right)^2\right)$, where the coefficient of the second order terms now involves $log(Q^2/s_0)$ (for details, see e.g. Ref. [21]). Integrating around $|s| = s_0$, for any analytic $w(s)$, the first surviving contribution is then the $\mathcal{O}\left(a(s_0)\right)^2$ logarithmic term. This logarithm, associated with the perturbative representation of $\alpha_s(Q^2)$, has maximum modulus precisely in the region (on either side of the cut on the time-like real axis) for which the perturbative representation is least reliable. FESR's associated with weight functions (such as $w(s) = s^k$) not suppressed near $s = s_0$ can thus have errors potentially much greater than those suggested by the naive estimate above. We illustrate this point for the case of the IV correlator below. For hadronic τ decay, however, phase space naturally produces a (double) zero of $w_\tau(s)$ at $s = m_\tau^2$, and this suppression of contributions from the region of the contour near the real timelike axis results in a very accurate FESR. We will see, in Section II, that other weight functions with zeros at $s = s_0$ also produce very reliable FESR's in the IV channel. We will then use such weights, and corresponding FESR's, in Section III, to studying the pseudoscalar isovector (PI) and strangeness $S = -1$ scalar (SS) channels (relevant to the extraction of the light quark mass combinations $m_u + m_d$ [10,22,23] and $m_s + m_u$ [24–26,23,11,27]).

II. LESSONS FROM HADRONIC τ DECAY

As noted above, FESR's involving weights, $w(s)$, with $w(s_0) \neq 0$ have significant *potential* uncertainties if local duality is not yet valid at scale s_0. To quantify this statement, consider the s^k-weighted FESR's for the IV channel. In Table I we list, as a function of s_0, the hadronic (I_k^{ex}) and OPE (I_k^{OPE}) sides of these sum rules,

$$I_k^{ex} \equiv \int_{4m_\pi^2}^{s_0} s^k \rho^{ex}(s) \, ds , \tag{5}$$

$$I_k^{(OPE)} \equiv -\frac{1}{2\pi i} \int_{|s|=s_0} s^k \Pi_{V,ud}^{(0+1)}(s) \, ds , \tag{6}$$

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for $k = 0, 1, 2, 3$. The hadronic side is evaluated using the spectral function, $\rho^{ex}(s)$, measured by ALEPH [18], while for the OPE side we employ the known OPE for $\Pi_{V,ud}^{(0+1)}(s)$ [14,16], together with (1) ALEPH values for $a(m_\tau^2)$ and the $D = 6$ condensate terms (from the non-strange decay analysis alone), (2) the gluon condensate of Ref. [28], (3) the GMO relation, $< 2m_\ell \bar{\ell}\ell > = -m_\pi^2 f_\pi^2$, (4) quark mass ratios from Chiral Perturbation Theory (ChPT) [29], (5) $0.7 < \left[< \bar{s}s > / < \bar{\ell}\ell > \right] < 1$, as in Refs. [24–26], and (6) four loop running, with contour improvement [15], for the perturbative contributions.

As seen from the Table, the errors in the s^k-weighted FESR's are significant, except near $s_0 \sim 2.8$ GeV^2, where the hadronic and OPE representations happen to cross. The worsening of agreement for s_0 above ~ 2.8 GeV^2 simply reflects the facts that (1) local duality is not valid for $s_0 \sim 3$ GeV^2 and (2) the problematic region of the circular part of the contour contributes significantly to the OPE side of the sum rule for $w(s) = s^k$, as one would expect. Note that the situation cannot be improved by taking "duality ratios" (ratios of such sum rules corresponding to different values of k) since, as pointed out in Ref. [30], if one insists on a match between the hadronic and OPE versions of such a duality ratio for s_0 values lying in some "duality window", then the spectral function is constrained to match (up to an undetermined overall multiplicative constant) that implied by the OPE for all s_0 in that window. If s_0 lies in the region of validity of local duality, this is not a problem, but if it does not (for example, if distinct resonances are still present), then contributions from the problematic part of the contour cannot have fully cancelled in the ratio.

The situation is much improved if we consider FESR's corresponding to weights with a zero at $s = s_0$. For reference we give, in Table II, the experimental (hadronic) and OPE sides of the FESR for the (double zero) combination $I_0 - 3I_2 + 2I_3$ relevant to hadronic τ decay (see also the discussion in Ref. [19]). As noted earlier, the match between the two sides is very good, even at low scales. We consider also the FESR's corresponding to Eq. (1), based on the weights $w_{k,k+1}(s) = (s/s_0)^k - (s/s_0)^{k+1}$ which have only a simple zero at $s = s_0$. Denoting the hadronic and OPE sides by $J_{k,k+1}^{(ex,OPE)}(s_0)$, we have

$$J_{k,k+1}^{(ex,OPE)}(s_0) = \frac{1}{s_0^k} I_k^{(ex,OPE)}(s_0) - \frac{1}{s_0^{k+1}} I_{k+1}^{(ex,OPE)}(s_0) \ . \tag{7}$$

The results for the cases $k = 0$ and $k = 1$ are given in Table II. Evidently, even a simple zero at $s = s_0$ is enough the suppress contributions from the problematic part of the contour sufficiently to produce sum rules that are very reliable, again even down to rather low scales.

The constraints on the hadronic spectral function obtained by combining the w_{01} and w_{12} sum rules are actually rather strong. To see this, note that a general linear combination of the two sum rules involves, for some constant A, a weight function proportional to

$$w(A, s) = \left(1 - \frac{s}{s_0} \right) \left(1 + A\frac{s}{s_0} \right) \ . \tag{8}$$

For $A < -1$, this weight has a second zero in the hadronic integration region which moves to lower s as A is decreased. To the left of this zero, the spectral function is weighted positively, to the right, negatively. Dialing the crossover location (by varying A) then places rather strong constraints on the spectral function, provided that (1) the OPE representation remains accurate over the range of A, s_0 values employed and (2) the errors on the OPE side are not unduly exacerbated by cancellations in forming the combination of the two sum rules. For the IV channel, it is straightforward to demonstrate that the errors are, indeed, not amplified, and that the resulting OPE and hadronic representations do, indeed, remain in excellent agreement, for $-9 \leq A \leq 9$ (over which range the weight function varies from having no second zero to having one below the ρ peak) and for a range of s_0 values extending well below m_τ^2. We do not display these facts explicitly since the central values for the OPE and hadronic representations follow from those already given in the Table. It is also worth stressing that, not only does the OPE representation match very well with the hadronic one based on experimental data, but that making a sum-of-resonances ansätz for the spectral function and fitting its parameters to the OPE representation produces a model spectral function in good agreement with the experimental one (for example, the resulting ρ decay constant differs from the experimental by less than the experimental error).

III. APPLICATIONS TO OTHER CHANNELS

We now consider FESR's for two channels of relevance to the extraction of the light quark masses. For the PI channel, unmeasured continuum contributions to the spectral function contribute roughly three-quarters of the hadronic side of the (integer power weighted) FESR which determines $(m_u + m_d)^2$, while for the SS channel, experimental constraints

exist only on the $K\pi$ portion of the spectral function. We employ FESR's based on the weights $w(A, s)$ in both these channels; in the former, to test the plausibility of the ansätz employed for the continuum part of the spectral function [10,22] and, in the latter, to test the viability of certain assumptions/approximations made in the earlier analyses.

For the PI channel, $m_u + m_d$ is extracted from sum rules for the correlator, $\Pi_5(q^2) = i \int d^4x \, e^{iq \cdot x} < 0| (\partial_\mu A^\mu(x) \partial_\mu A^\mu(0)) |0 >$, where A^μ is the isovector axial vector current. With $\rho_5(s)$ the corresponding spectral function, one has, using $w(s) = s^k$ [10,22],

$$\int_0^{s_0} ds \, s^k \, \rho_5(s) = \frac{3}{8\pi^2} [m_u(s_0) + m_d(s_0)]^2 \frac{s_0^{k+2}}{k+2} [1 + R_{k+1}(s_0) + D_k(s_0)] + \delta_{k,-1} \Pi_5(0) \qquad (9)$$

where $R_{k+1}(s_0)$ (the notation is that of Ref. [10]) contain the perturbative corrections, D_k the contributions from higher dimension operators, and $\Pi_5(0)$ is determined by f_π, m_π and the combination $2L_8^r - H_2^r$ of fourth order ChPT low energy constants (LEC's) (see Ref. [31]). The analysis of Refs. [10,22] (BPR) proceeds by (1) adjusting the relative strength of $\pi(1300)$ and $\pi(1800)$ contributions to a continuum spectral ansatz using the $k=0$ to $k=1$ duality ratio, (2) fixing the overall scale of this ansätz by normalizing the sum of the resonance tails to the leading order ChPT expression for $\rho_5(s)$ at continuum threshold, (3) (with $\rho_5(s)$ so fixed) using the $k=0$ and $k=-1$ sum rules to extract $m_u + m_d$ and $2L_8^r - H_2^r$, respectively. A number of possible problems exist with this analysis. First (see Ref. [30]) there are potential dangers in the overall normalization prescription, associated with the fact that continuum threshold is rather far from the resonance peak locations. Second, the value of the LEC combination obtained implies an unusual value for the light quark condensate ratios. (The combination $2L_8^r - H_2^r$ is related to $2L_8^r + H_2^r$, which controls flavor breaking in the condensate ratios [31]. With standard values for L_8^r [31], the BPR $2L_8^r - H_2^r$ value corresponds to $< \bar{s}s > / < \bar{u}u >= 1.30 \pm 0.33$.) Finally, the presence of the $\pi(1800)$ signals that one is not in the region of local duality, and hence that non-negligible errors may be present in s^k-weighted FESR's (and duality ratios thereof). We can investigate this latter question by considering those additional constraints on the BPR continuum spectral ansätz obtained from the $w(A, s)$ family of FESR's. As input to the OPE side we use the latest ALEPH determination of $a(m_\tau^2)$, the condensate values employed in Refs. [10,22], the most recent value of Ref. [22] for $m_u + m_d$, and the four-loop contour-improved version of the perturbative contributions. Apart from the small decrease in the ALEPH value of $a(m_\tau^2)$ between 1997 and 1998 the input to the OPE analysis is, therefore, identical to that of Refs. [10,22]. To be specific in tabulating results, we have employed, on the hadronic side, the updated continuum ansätz of Ref. [22] (the situation is not improved if one uses instead any of the earlier ansatze of Ref. [10]). In Table III we present the hadronic (had) and OPE sides of the resulting sum rules for s_0 in the BPR duality window and the range $0 \leq A \leq 9$. Note that the best duality match from Ref. [22] corresponds to $s_0 = 2.0 \sim 2.4$ GeV^2. In all cases, the known π pole contribution has been subtracted from both sides of the sum rule; the results thus provide a direct test of the continuum spectral ansätz. While one cannot guarantee that the $w(A, s)$ sum rules will work as well in the PI as in the IV channel, there are clear physical grounds for expecting them to be more reliable than those based on the weights $w(s) = s^k$. If one were actually in the region of local duality, and had a good approximation to the physical continuum spectral function, then of course the two methods would be compatible. Since they are not, we conclude that either the OPE is simply not well enough converged to provide a reasonable representation of the correlator away from the timelike real axis, for the s_0 values considered (in which case the whole analysis collapses as a method for extracting $m_u + m_d$), or the BPR continuum spectral ansatz, and hence the estimate of $m_u + m_d$ based on it, is unreliable. The former seems implausible (the contour-improved series, particularly at the somewhat larger scales shown in the table, appears rather well-behaved) though it cannot be rigorously ruled out.

Let us turn now to the SS channel. Here one replaces A^μ above with the $S = -1$ vector current $\bar{s}\gamma^\mu u$, and obtains sum rules involving the $S = -1$ scalar correlator, $\Pi(s)$. The analyses of Refs. [24,25] (JM/CPS) and [26] (CFNP) employ the conventional SVZ method. In the former, the Omnes representation, together with experimental K_{e3} and $K\pi$ phase shift data, is used to fix the timelike $K\pi$ scalar form factor at continuum threshold $s = (m_K + m_\pi)^2$. A sum-of-resonances ansätz for the spectral function, $\rho(s)$, normalized to this value, is then employed on the hadronic side. In contrast, CFNP employ the Omnes representation also above threshold in order to obtain the $K\pi$ contribution to $\rho(s)$ purely in terms of experimental data. This improves the low-s behavior of the spectral function (which shows considerable distortion associated with the attractive $I = 1/2$ s-wave $K\pi$ interaction). Unresolved issues for the CFNP analysis include (1) the size of spectral contributions associated with neglected higher multiplicity states, (2) sensitivity to the assumption that the $K\pi$ phase is constant at its asymptotic value (π) beyond the highest s ($\sim (1.7$ GeV$)^2$), for which it is known experimentally, and (3) the failure to find a stability window unless the continuum threshold is allowed to lie significantly above the region of significant $K\pi$ spectral contributions (leaving an unphysical region of size $1 - 2$ GeV^2 with essentially no spectral strength in the resulting spectral model). We investigate the CFNP ansätz (which retains only the $K\pi$ portion of $\rho(s)$, obtained as described above) by studying again the $w(A, s)$ family of FESR's. In Table IV are displayed the OPE and hadronic sides, as a function of s_0 and A, for both the JM/CPS and CFNP spectral ansatze. In each case, the value of $m_s(1$ GeV$^2)$ extracted by the earlier authors has been used

as input on the OPE side. For the CFNP case, the original authors quote a range of values; to be specific, we have chosen that value from this range, $m_s = 155$ MeV, which produces a match of the OPE and hadronic sides of the sum rule for $s_0 = 4$ GeV^2 and $A = 0$. Comparing the two sides at other values of s_0, A then provides a test of the quality of the spectral ansätz. The agreement is obviously much better for the CFNP ansätz than in the other two cases, though amenable to some further improvement. Note that the most obvious improvement, namely adding additional spectral strength in the vicinity of the $K_0^*(1950)$ to account for contributions of multiparticle states (the $K\pi$ branching ratio of the $K_0^*(1950)$ is $52 \pm 8 \pm 12\%$ [32]), actually somewhat worsens the agreement between the OPE and hadronic sides, suggesting that modifications of the spectrum at lower s may be required.

IV. SUMMARY

We have shown that continuous families of FESR's can be used to place constraints on hadronic spectral functions and that, based both on qualitative physical arguments and a study of the IV channel, these constraints can be expected to be more reliable than those based on FESR's with integer power weights. The method is complementary to conventional Borel transformed (SVZ) treatments in that it involves weights that do not suppress (and for some A actually enhance) contributions from the higher s portion of the spectrum.

Relevant to attempts to extract the light quark masses, the method has been shown to produce constraints on the continuum portions of hadronic spectral functions not exposed by previous sum rule treatments. For the PI channel, one finds that either the use of the OPE representation is not justified at the scales considered, or existing spectral ansatze must be modified significantly to produce an acceptable match to the known OPE representation. The need for (albeit less significant) modifications to the CFNP spectral model in the SS channel has also been illustrated. We conclude that the question of the values of the light quark masses is still open, particularly for $m_u + m_d$, and that further investigations using the method explored here may help in clarifying the situation.

ACKNOWLEDGMENTS

TABLE I. Integer power weighted OPE and hadronic integrals, $I_k^{ex(OPE)}$ for the IV channel. All notation as defined in the text. Units of $I_k^{ex(OPE)}$ are GeV$^{2(k+1)}$, and of s_0 are GeV^2.

s_0	ex/OPE	$I_0^{ex/OPE}$	$I_1^{ex/OPE}$	$I_2^{ex/OPE}$	$I_3^{ex/OPE}$
2.0	ex	.0559±.0006	.0483±.0007	.0539±.0010	.0736±.0017
	OPE	.0612±.0013	.0584±.0009	.0746±.0010	.110±.001
2.2	ex	.0620±.0008	.0610±.0010	.0804±.0018	.129±.004
	OPE	.0667±.0013	.0704±.0009	.0998±.0012	.163±.002
2.4	ex	.0692±.0009	.0773±.0014	.117±.003	.213±.006
	OPE	.0722±.0012	.0834±.0010	.130±.001	.233±.002
2.6	ex	.0765±.0010	.0953±.0018	.162±.004	.324±.009
	OPE	.0777±.0012	.0976±.0011	.165±.002	.322±.003
2.8	ex	.0841±.0013	.116±.003	.216±.007	.470±.018
	OPE	.0832±.0012	.113±.001	.207±.002	.433±.004
3.0	ex	.0924±.0025	.139±.007	.285±.019	.668±.055
	OPE	.0888±.0012	.129±.001	.254±.002	.571±.005
3.1	ex	.0970±.0041	.154±.012	.328±.036	.797±.108

	OPE	.0915±.0012	.138±.001	.281±.002	.651±.005

TABLE II. Vector isovector channel FESR's for weight functions with a zero at $s = s_0$. Columns 2 and 3 contain the hadronic (ex) and OPE sides, $J_\tau^{ex,OPE}$, of the sum rule corresponding to the double zero weight function occurring in hadronic τ decay. Columns 4 and 5, and 6 and 7, contain the hadronic and OPE sides, $J_{01}^{ex,OPE}$, $J_{12}^{ex,OPE}$, of the sum rules corresponding to the weight functions, $w_{01}(s)$, $w_{12}(s)$, having a simple zero at $s = s_0$. All notation is as defined in the text. Units of s_0, $J_\tau^{ex(OPE)}$ and $J_{k-1,k}^{ex(OPE)}$ are GeV^2.

s_0	J_τ^{ex}	J_τ^{OPE}	J_{01}^{ex}	J_{01}^{OPE}	J_{12}^{ex}	J_{12}^{OPE}
2.0	.0339±.0005	.0337±.0010	.0318±.0005	.0323±.0009	.0107±.0002	.0103±.0003
2.2	.0364±.0005	.0363±.0010	.0343±.0005	.0350±.0009	.0111±.0002	.0111±.0002
2.4	.0389±.0005	.0389±.0009	.0370±.0005	.0378±.0008	.0118±.0002	.0120±.0002
2.6	.0414±.0005	.0413±.0009	.0398±.0005	.0406±.0008	.0127±.0002	.0128±.0002
2.8	.0440±.0005	.0440±.0009	.0428±.0005	.0433±.0008	.0137±.0002	.0137±.0002
3.0	.0467±.0005	.0466±.0009	.0458±.0005	.0461±.0008	.0148±.0002	.0146±.0002
3.1	.0481±.0005	.0481±.0009	.0475±.0005	.0475±.0008	.0154±.0003	.0150±.0002

TABLE III. Comparison of the OPE and hadronic (had) sides of the FESR's corresponding to the weight family $w(A,s)$ defined in the text, for the pseudoscalar isovector channel. Units of s_0 are GeV^2 and of the w-weighted integrals, 10^{-6} GeV^6. The known pion pole contribution has been subtracted from both sides of the sum rule.

s_0	had/OPE	$A = 0$	$A = 3$	$A = 6$	$A = 9$
2.0	had	2.18	6.76	11.3	15.9
	OPE	4.42	14.5	25.6	34.7
2.4	had	3.98	11.8	19.5	27.3
	OPE	6.94	20.3	33.7	47.0
2.8	had	5.68	15.8	25.9	35.9
	OPE	9.68	26.6	43.6	60.5
3.2	had	7.22	19.0	30.8	42.6
	OPE	12.6	33.4	54.3	75.1

TABLE IV. Comparison of OPE and hadronic sides of the $w(A, s)$ family of FESR's for the strangeness $S = -1$ scalar channel and for both the JM/CPS and CFNP ansatze for the hadronic spectral function. The OPE results are obtained using the values of m_s determined in the earlier (SVZ-style) analyses. Units of s_0 are GeV^2, and of the hadronic/OPE integrals, 10^{-3} GeV^6.

Case	s_0	had/OPE	$A = 0$	$A = 3$	$A = 6$	$A = 9$
JM/CPS	3.2	OPE	4.39	9.45	14.5	19.6
		had	5.76	15.5	25.3	35.1
	3.4	OPE	4.77	10.3	15.9	21.4
		had	6.37	16.7	27.1	37.5
	3.6	OPE	5.15	11.2	17.3	23.3
		had	6.94	17.8	28.7	39.6
	3.8	OPE	5.56	12.1	18.7	25.3
		had	7.50	18.9	30.2	41.6
	4.0	OPE	5.97	13.1	20.2	27.3
		had	8.05	19.9	31.7	43.5
CFNP	3.2	OPE	2.51	5.35	8.20	11.0
		had	2.65	6.56	10.5	14.4
	3.4	OPE	2.72	5.84	8.96	12.1
		had	2.85	6.90	11.0	15.0
	3.6	OPE	2.94	6.34	9.75	13.2
		had	3.04	7.18	11.3	15.5
	3.8	OPE	3.16	6.86	10.6	14.3
		had	3.21	7.41	11.6	15.6
	4.0	OPE	3.40	7.40	11.4	15.4
		had	3.37	7.40	11.8	16.1

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